Calculus of inverse trigonometric functions

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Review

Recall from Extension 1 that we can define the **inverse** of a real-valued function $f(x)$ once we restrict f to a domain on which it is **one-to-one**. Aside from the exponential and the logarithm, the inverse trigonometric functions form a second class of inverse functions that is essential for the HSC. Intuitively, inverse functions should "undo" each other, for example

$$e^{\ln x} = \ln(e^x) = x \qquad \text{for all } x > 0$$

However, one has to be very careful when appealing to this intuition, because it actually depends on the choice of domain. (See part (ii) of Exercise 1).

Reminders

- The **inverse sine** $\sin^{-1}(x)$ is defined by restricting the function $\sin(x)$ to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
- The **inverse cosine** $\cos^{-1}(x)$ is defined by restricting the function $\cos(x)$ to the domain $[0, \pi]$.
- The **inverse tangent** $\tan^{-1}(x)$ is defined by restricting the function $\tan(x)$ to the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Exercise 1.

(i) State the domain and range of $\sin^{-1}(x)$.

(ii) Hence sketch the functions

$$f(x) = \sin(\sin^{-1}(x)) \qquad \text{and} \qquad f(x) = \sin^{-1}(\sin(x))$$

in their natural domains (**Warning**: it is definitely *wrong* to draw just the straight line $f(x) = x$). You may use graphing software such as Desmos to check your answer.

(iii) State the domain and range for the $\cos^{-1}$ and $\tan^{-1}$ functions, and then sketch them.

The calculus of inverse trignometric functions is an important part of the Extension 2 course, and you will find that derivatives and integrals involving these functions turn up in all kinds of applications, particularly in mechanics. In these notes, we will prove some standard calculus results for inverse trigonometric functions.

Differentiation

How can we find the derivative of the inverse trigonometric functions? As with many things in the more advanced study of calculus, the **chain rule** is our friend. In fact, our method will work in general, not only for inverses of trigonometric functions.

To avoid confusion, let's give the inverse function a different letter, say $g(x) = f^{-1}(x)$. Assuming we are in the correct domain, we can write

$$f(g(x)) = x$$

What happens when we differentiate both sides of this equation? On the right hand side, obviously we get 1. On the left hand side, we use the chain rule to see that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Therefore $f'(g(x))g'(x) = 1$, which implies

$$\frac{d}{dx}(f^{-1}(x)) = g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))}$$

We have proved the following:

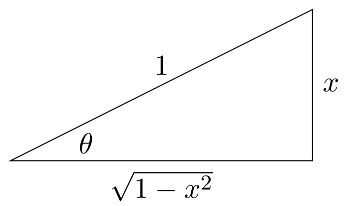
Important result

$$\boxed{\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}}$$

Let's see what we discover when we apply our important result to the trigonometric functions. Let $f(x) = \sin x$, then $f'(x) = \cos x$. Using the formula above, we obtain

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))}$$

At first, this looks silly, because of the weird expression $\cos(\sin^{-1}(x))$. However, this is easy to fix. Let $\theta = \sin^{-1} x$, which means $\sin \theta = x$.



Drawing the appropriate triangle, we see that $\cos \theta = \sqrt{1 - x^2}$. We have discovered the derivative of inverse sine!

$$\boxed{\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}}$$

Exercise 2. Use the above method to show that

$$\boxed{\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1 - x^2}}}$$

and

$$\boxed{\frac{d}{dx} \tan^{-1}(x) = \frac{1}{x^2 + 1}}$$

For inverse tan, you will need to recall that $\frac{d}{dx} \tan x = \sec^2(x)$.

Exercise 3.

(i) Using differentiation, prove that

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$$

(ii) Using calculus to prove the above identity is overkill. See if you can show that it is true using only basic trigonometry.

Exercise 4 (Chain rule practice)**.** Differentiate the following functions (do not worry about the domains):

(i) $e^{\sin^{-1}(x)}$

(ii) $\sin^{-1}(e^x)$

(iii) $\tan^{-1}(x^2)$

(iv) $(\tan^{-1}(x))^2$

(v) $\sqrt{\cos^{-1}(x)}$

(vi) $\dfrac{1}{\sqrt{1-(\sin^{-1}x)^2}}$

Integration

Since we know the derivatives of the inverse trigonometric functions now, we immediately obtain the following integrals:

$$\boxed{\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1}(x) + C \\ \int \frac{1}{x^2+1}\,dx &= \tan^{-1}(x) + C\end{aligned}}$$

Exercise 5. From the derivative of $\cos^{-1}(x)$, we get

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + C$$

but on the other hand,

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}\,dx = -\sin^{-1}(x) + C$$

Explain why there is no contradiction.

However, in practice, we are not always so lucky to encounter such simple integrals. Far more common are integrals of the form

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx \quad \text{and} \quad \int \frac{1}{a^2+x^2}\,dx$$

It is reasonable to expect that they integrate to inverse trigonometric functions. The question is, what is the effect of the constant a when $a \neq 1$?

We can investigate in the following way. For convenience, suppose $a > 0$, and consider the function $\frac{1}{\sqrt{a^2-x^2}}$. We can factor out the a^2:

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{a\sqrt{1-(\frac{x^2}{a^2})}}$$

Now we make the substitution $u = \frac{x}{a}$, so that $a\,du = dx$. Then

$$\begin{aligned}\int \frac{1}{a\sqrt{1-(\frac{x^2}{a^2})}}\,dx &= \frac{1}{a}\int \frac{1}{\sqrt{1-u^2}}a\,du \\ &= \int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \sin^{-1}(u) + C = \sin^{-1}(\tfrac{x}{a}) + C\end{aligned}$$

Thus we have obtained the integral

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C}$$

In a similar way, we can prove that

$$\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

I will leave the details for you to fill in. A common error is to forget the factor of $1/a$ in front of the $\tan^{-1}$! (If you carried out the proof correctly, you will understand why it is there).

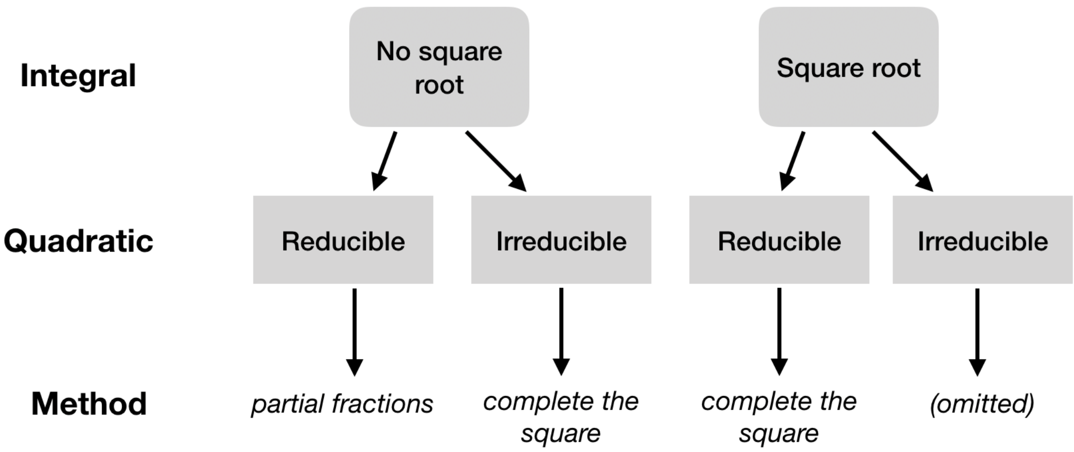
Integrals that require completing the square

We will now look at integrals of the form

$$\frac{1}{\sqrt{\text{quadratic}}} \quad \text{or} \quad \frac{1}{\text{quadratic}}$$

We do not discuss the case where the quadratic has a double root (which is much easier and not relevant here). Under this assumption, there are two cases remaining:

- **Irreducible**: this means that it cannot be factored over the real numbers.
- **Reducible**: this means we can write the quadratic as the product of distinct linear factors $(Ax - B)(Cx - D)$.



Hence we have four possible types of integrals. All the essential information is shown in the flow chart above.

Example 1. Find the integral

$$\int \frac{1}{\sqrt{6x - x^2}}\,dx$$

Solution: The quadratic is clearly reducible. Hence we complete the square: $6x - x^2 = 9 - (x-3)^2$. This gives an integral leading to an inverse sine function:

$$\int \frac{1}{\sqrt{6x - x^2}}\,dx = \int \frac{1}{\sqrt{9 - (x-3)^2}}\,dx = \sin^{-1}\left(\frac{x-3}{3}\right) + C$$

Example 2. Find the integral

$$\int \frac{1}{x^2 + 6x + 13}\,dx$$

Solution: The quadratic is irreducible (the discriminant is $6^2 - (4 \times 13) = -16 < 0$). Hence we can complete the square: $x^2 + 6x + 13 = (x+3)^2 + 4$. This gives an integral leading to an inverse tangent function:

$$\int \frac{1}{x^2 + 6x + 13}\,dx = \int \frac{1}{(x+3)^2 + 4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

Example 3 (Slightly more challenging)**.** Find the integral

$$\int \frac{x}{\sqrt{4 + 2x^2 - x^4}}\,dx$$

Solution: It seems like a good idea to use the substitution $u = x^2$. This gives $du = 2x\,dx$, hence $x\,dx = \frac{1}{2}du$. The integral now becomes

$$\int \frac{x}{\sqrt{4+2x^2-x^4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{4+2u-u^2}}\,du$$

Checking the discriminant of the quadratic, we get $2^2 - 4(-1)(4) = 20 > 0$, so it is reducible. We complete the square to obtain $4 + 2u - u^2 = 5 - (u-1)^2$, and thus

$$\begin{aligned}\frac{1}{2}\int \frac{1}{\sqrt{4+2u-u^2}}\,du &= \frac{1}{2}\int \frac{1}{\sqrt{5-(u-1)^2}}\,du \\ &= \frac{1}{2}\sin^{-1}\left(\frac{u-1}{\sqrt{5}}\right) + C \\ &= \frac{1}{2}\sin^{-1}\left(\frac{x^2-1}{\sqrt{5}}\right) + C\end{aligned}$$

Here is a summary of the main points.

(a) **No square root**

- If the quadratic is irreducible, we can *complete the square*, leading to an inverse tangent function after integrating.
- If the quadratic is reducible, we need to use the method of *partial fractions*. (This does not lead to an inverse trigonometric function).

(b) **With square root**

- If the quadratic is irreducible, this leads to a tricky case (see Appendix).
- If the quadratic is reducible, then we can *complete the square*, leading to an inverse sine function after integrating.

Appendix

The case where we have an irreducible quadratic inside a square root is tricky. We give a typical example below.

Example 4. Let $a > 0$. Find the integral

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

Solution: Let $x = a\tan\theta$. Then $dx = a\sec^2\theta\,d\theta$, so the integral becomes

$$\begin{aligned}\int \frac{1}{\sqrt{a^2+a^2\tan^2\theta}}a\sec^2\theta\,d\theta &= \int \frac{a\sec^2\theta}{a\sec\theta}\,d\theta \qquad \text{(recall that } 1+\tan^2\theta = \sec^2\theta\text{)} \\ &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C\end{aligned}$$

Now we restore the original variable. If $x = a\tan\theta$, then $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+(x/a)^2}$. Hence

$$\begin{aligned}\int \frac{1}{\sqrt{a^2+x^2}}\,dx &= \ln|\sqrt{1+(x/a)^2} + x/a| + C \\ &= \ln(x + \sqrt{a^2+x^2}) + C\end{aligned}$$