Derivatives of trigonometric functions

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Review of differentiation from first principles

In these notes, we will discover the derivative of $\sin x$ using a classic geometric proof. Recall that a function $f(x)$ defined on a domain $D \subset \mathbb{R}$ is said to be *differentiable* at a point $x_0 \in D$ if the following limit exists:

$$\boxed{\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}} \tag{1}$$

In this case, the *derivative* of f at x_0 is denoted by

$$f'(x_0) \qquad \text{or} \qquad \left.\frac{df}{dx}\right|_{x=x_0}$$

and is given by the above limiting value.

When we calculate derivatives using the definition, this is called *differentiation from first principles*. As a simple example, we can consider $f(x) = x^2$ with domain $D = \mathbb{R}$. You may recall from your previous study that $f'(x) = 2x$, but we can prove this rigorously using first principles. Let x_0 be an arbitrary real number. Then

$$\begin{aligned} f'(x_0) &= \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} \qquad \text{(by definition!)} \\ &= \lim_{h \to 0} \frac{2x_0 h + h^2}{h} \\ &= \lim_{h \to 0} (2x_0 + h) = 2x_0 \end{aligned}$$

Since x_0 was arbitrary, we therefore have $f'(x) = 2x$ for all real numbers $x \in \mathbb{R}$, as we knew already.

Of course, first principles is *not* usually the preferred method for calculating derivatives, because in general the limit will be extremely difficult to evaluate. However, first principles is sometimes the only way to prove a theoretical result. We will now prove that

$$\boxed{\frac{d}{dx} \sin x = \cos x} \tag{2}$$

You may have already learned this fact without proof. However, since we intend to *prove* the result, it is best to forget temporarily that you have learned it before reading on!

The derivative of $f(x) = \sin x$ via geometry

As a first step, let us write down the derivative of $\sin x$ using the first principles definition:

$$f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$$

There does not seem to be much more we can do, except to expand the term $\sin(x+h)$ using the compound angle formula:

$$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

Inserting this identity back into the limit expression and simplifying, we obtain

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \sin(x) \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} \end{aligned}$$

Thus we have to compute the following limits

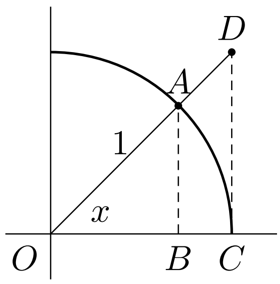
$$\lim_{h\to 0} \frac{\sin(h)}{h} \qquad \lim_{h\to 0} \frac{\cos(h)-1}{h} \tag{$\star$}$$

As it turns out, the first one is the tricky part, and the second limit is a consequence of the first. To compute the first limit, we give a classic geometric proof.

Proposition 1.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1} \tag{3}$$

Proof. We first assume $x > 0$. Consider the following picture of the unit circle in the first quadrant:



Clearly, we have the following inequalities:

$$\text{length of } AB < \text{length of arc } AC < \text{length of } DC$$

From basic trigonometry, we have

$$|AB| = \sin x \qquad \text{and} \qquad |DC| = \tan x$$

and moreover, recalling the definition of radians, we have

$$\text{length of arc } AC = x$$

Therefore

$$\sin x < x < \tan x = \frac{\sin x}{\cos x}$$

Now $\sin x > 0$ for $x > 0$, so we can divide through by $\sin x$ in the above inequalities to obtain

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

Hence

$$\cos x < \frac{\sin x}{x} < 1$$

Notice that $\cos x \to 1$ as $x \to 0$. By the *squeeze law* for limits, we conclude

$$\lim_{x\to 0} \cos x = 1 \leq \lim_{x\to 0} \frac{\sin x}{x} \leq 1$$

i.e. $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

In the above calculations, we started with $x > 0$ and then let x approach 0 from above. We also need to check that we obtain the same limit when $x < 0$ and x approaches 0 from below. Fortunately, we do not need to repeat the above arguments. Notice that the last step only required knowing that $\cos x \to 1$ as $x \to 0$. This is true for both $x > 0$ approaching 0 from above, and $x < 0$ approaching 0 from below. Hence

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

concluding the proof. □

The limit we have just computed is an important one, so it is worth remembering the result, even if you don't remember the details of the proof.

Now we show how to obtain the second limit in ($\star$) using the result above.

Corollary 1.

$$\boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0} \tag{4}$$

Proof. The proof involves a simple algebraic trick. We write

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

Now

$$\frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}$$

and we know how to compute the limits that appear on the right hand side of the above equation! We have

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

by Proposition 1, and

$$\lim_{x\to 0} \frac{\sin x}{\cos x + 1} = \frac{\sin(0)}{\cos(0) + 1} = 0$$

(Notice that we can simply plug in $x = 0$ if the function is continuous and the value of the function is defined at that point). Combining everything, we can conclude

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = -\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{\cos x + 1} = 1 \times 0 = 0$$

as claimed. □

Finally, we put everything together and finish our original calculation!

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} &= \sin(x) \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h\to 0} \frac{\sin(h)}{h} \\ &= (\sin x) \times 0 + (\cos x) \times 1 \\ &= \cos x \end{aligned}$$

Derivatives of other trigonometric functions

Now that we know the derivative of $\sin x$, we can obtain the derivatives of all the other trigonometric functions very easily by exploiting their close relationships.

1. **Cosine**: Remember that "co" stands for "complementary angle" in trigonometry. The cosine of an angle x is the sine of the complementary angle:

$$\cos(x) = \sin(\tfrac{\pi}{2} - x) \quad \text{and} \quad \sin(x) = \cos(\tfrac{\pi}{2} - x)$$

Thus by the chain rule, we have

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\sin(\tfrac{\pi}{2} - x) = -\cos(\tfrac{\pi}{2} - x) = -\sin x$$

2. **Tangent and cotangent**: From the definition $\tan x = \frac{\sin x}{\cos x}$, we can use the quotient rule and the derivatives of sin and cos to calculate the derivative of $\tan x$:

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

To find the derivative of $\cot x$, we could either use the quotient rule, since $\cot x = \frac{\cos x}{\sin x}$, or we could use complementary angles and the chain rule. In any case, you should have no problem verifying that

$$\frac{d}{dx}\cot x = \frac{-1}{\sin^2 x} = -\csc^2 x$$

3. **Secant and cosecant**: We have $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$. We leave it as an easy exercise to verify the following:

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Summary of important results

- The following limit is useful to remember (especially for Extension 2 students, less important for Extension 1):

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

- The following derivatives are *essential* to remember:

$$\boxed{\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x}$$

- These derivatives do not appear so often in HSC exams, and in any case they can be easily deduced using a combination of complementary angle definitions, chain rule and quotient rule:

$$\boxed{\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x \qquad \frac{d}{dx}\cot x = \csc^2 x}$$